Using Random Set Theory to solve Challenge Problem B as proposed by the Epistemic Uncertainty Project, Sandia National Laboratories

By

Fulvio Tonon
University of Utah
Geology & Geophysics
135 South 1460 East, Room 719
Salt Lake City, UT 84112
Phone: +1-801-585-0491
Email: tonon@geophys.utah.edu

Abstract. The Epistemic Uncertainty Project of Sandia National Laboratories (NM, USA) proposed two challenge problems intended to assess the applicability and the relevant merits of modern mathematical theories of uncertainty in reliability engineering and risk analyses. This paper proposes a solution to Problem B: the response of a mechanical system with uncertain parameters. Random Set Theory is used to cope with both imprecision and dissonance affecting the available information. Imprecision results in an envelope of CDFs of the system response bounded by an Upper CDF and a Lower CDF. Different types of parameter dicretizations are introduced. It is shown that: (i) when the system response presents extrema in the range of parameters considered, it is better to increase the fineness of the discretization than to invoke a global optimization tool; (ii) the response expectation differed by less than 0.5% when the number of function calls was increased 15.7 times; (iii) larger differences (4-5%) were obtained for the lower tails of the CDFs of the response. Further research is necessary to investigate (i) parameter discretizations aimed at increasing the accuracy of the CDFs lower tails; (ii) the role of correlation in information composition.

1. Introduction

The Epistemic Uncertainty Project was formed at Sandia National Laboratories (NM, USA) to investigate the applicability and usefulness of some of the modern mathematical theories of uncertainty in the context of epistemic uncertainty (lack of knowledge), as opposed to aleatory uncertainty (random processes).

In order to assess the applicability and relevant merits of these theories in reliability engineering and risk analyses, two challenge problems were proposed by Oberkampf *et al.* (2001). This paper proposes a solution to Challenge Problem B, which deals with the response of a mass-spring-damper system acted on by a harmonic forcing. The assigned information on the parameters controlling the system response is affected either by dissonance or by imprecision, or by both dissonance and imprecision as defined hereafter (Klir, 1989). By dissonance we mean "the disagreement resulting from the attempt to classify an element of a given universal set into two or more disjoint subsets of interest under total or partial ignorance regarding relevant characteristics of the element". By imprecision (or non-specificity) we mean "the variety of alternatives that in a given situation are left unspecified'.

As shown by Walley (1991): (i) the amount of information concerning an event is more closely related to its degree of imprecision, rather than to its degree of dissonance; (ii) imprecision can be reconciled with coherence and rationality and is necessary when reasoning in situations of little information.

Problem B is detailed in Section 2, and presents two main issues:

1) How to model the uncertainty affecting the information available on each parameter.
2) How to propagate uncertainty to the system response, which must be treated as a "black box" (these are actually two sub-issues).

In the approach presented herein, Issue 1 is dealt with by modeling each parameter by means of a Random Set; the basic definition of a Random Set is given in the Appendix. Section 3 illustrates this approach.

Issue 2 is dealt with in Section 4 by means of the Extension Principle for Random Sets; for computational reasons (the system response must be treated as a black box), the random set constraining each parameter is discretized, so that the focal elements are finite in number.

Both the uncertainty modeling and the computational aspects of the proposed procedure are discussed. In order to allow easy comparison with procedures used by other authors, numerical values are given for the obtained results.

2. Problem statement

Oberkampf *et al.* (2001) consider a linear mass-spring-damper system subjected to a harmonic external force as depicted in Figure 1. Let:

- m be the mass.
- k be the spring stiffness.
- c be the damper viscosity.
- ω be the circular frequency of the applied force.

The four parameters c, k, m, and ω are independent, i.e. knowledge about the value of one parameter implies nothing about the value of the other. The information on each parameter is as follows:

- Parameter m: as shown in Figure 2, m is given by a triangular probability distribution defined on the interval $[m_{min}, m_{max}] = [10, 12]$, with a mode $m_{mod} = 11$.
- Parameter k: information on k is given by 3 independent and equally credible sources of information. Each source agrees that k is given by a triangular distribution. However, each

source specifies closed intervals, Min_i, Mod_i, and Max_i of possible values of the minimum, k_{min}, mode, k_{mode}, and maximum value, k_{max}, respectively, as follows (Figures 3a-3c):

Min_1 = [90, 100]; Mod_1 = [150, 160]; Max_1 = [200, 210]

Min_2 = [80, 110]; Mod_2 = [140, 170]; Max_2 = [200, 220]

Min_3 = [60, 120]; Mod_3 = [120, 180]; Max_3 = [190, 230]

The relationship among these values is:

$$k_{min} \leq k_{mode} \leq k_{max} \text{ and } k_{min} < k_{max} \tag{1}$$

Finally, Min_i, Mod_i, and Max_i are consistent collections of intervals, i.e.:

$$\bigcap_{i=1}^{3} Min_i \neq \varnothing, \bigcap_{i=1}^{3} Mod_i \neq \varnothing, \bigcap_{i=1}^{3} Max_i \neq \varnothing \tag{2}$$

- Parameter c: information on c is given by 3 independent and equally credible sources of information. As illustrated in Figure 4, each source specifies a closed interval C_j of possible values of c as follows:

  C_1 = [5, 10]

  C_2 = [15, 20]

  C_3 = [25, 25]

  $\{C_j\}$ is an arbitrary collection of intervals, i.e. there is no assumed overlap or relationship among any of the intervals of the collection.

- Parameter w: as depicted in Figure 5, information on w is given by a triangular probability distribution whose minimum value, w_{min} is the interval [2, 2.3], whose mode value, w_{mod} is the interval [2.5, 2.7], and whose maximum value, w_{max} is the interval [3.0, 3.5]. The relationship among these values is:

$$w_{min} \leq w_{mode} \leq w_{max} \text{ and } w_{min} < w_{max} \tag{3}$$

The steady state magnification factor D_s is defined as the ratio of the amplitude of the steady-state response of the system to the static displacement of the system (Eq. 26 in Oberkampf *et al.*, 2001):

$$D_s = \frac{k}{\sqrt{(k - mw)^2 + (cw)^2}} \tag{4}$$

We notice that the external load amplitude does not appear in Eq. (4). The objective of this problem is to quantify the uncertainty in the steady-state magnification factor D_s, given the stated information for the problem.

3. Modeling the uncertainty affecting the parameters

In the following, two discretizations will be used, namely a coarser one (Discretization A), and a finer one (Discretization B).

3.1 Parameter m

Parameter *m* is assigned by means of its probability density function, and therefore is only affected by dissonance. As shown in Tonon *et al.* (2000b), when all parameters of a given model are assigned by means of their PDFs (or their joint PDF), Random Set Theory can be used to efficiently bracket the results of Monte Carlo simulations. As far as parameter *m* is concerned, its discretization into a random set is performed for computational purposes only.

The procedure is as follows (Tonon *et al.* 2000b):

1) Discretize intervals $[m_{min}, m_{mod}]$ and $[m_{mod}, m_{max}]$ into n_1 and n_2 subintervals $A_{m,i} = [a_i, b_i]$, respectively. Each subinterval $A_{m,i}$ is treated as a focal element.

2) Let $p(m)$ be the PDF of m, and $F_m(m)$ the CDF of m. Calculate the basic probability assignment $M_m(A_{m,i})$ for focal element $A_{m,i}$ as:

$$M_m(A_{m,i}) = \int_{A_{m,i}} p(m)dm = F_m(b_i) - F_m(a_i) \tag{5}$$

With the assigned numerical values, one obtains:

$$M_m(A_{m,i}) = \frac{1}{2}\left[(b_i - 20)b_i - (a_i - 20)a_i\right] \quad if \quad A_{m,i} \in [m_{\min}, m_{\mathrm{mod}}] \tag{6a}$$

$$M_m(A_{m,i}) = \frac{1}{2}\left[(-b_i + 24)b_i + (a_i - 24)a_i\right] \quad if \quad A_{m,i} \in [m_{\mathrm{mod}}, m_{\max}] \tag{6b}$$

Tables 1.a and 1.b give the random sets of parameter m for Discretizations A ($n_1 = n_2 = 5$) and B ($n_1 = n_2 = 10$), respectively. Figures 6.a and 6.b depict the CDF of m along with the Lower Cumulative and Upper Cumulative Distribution Functions calculated by means of Eqs. (A.7a-b) for Discretizations A and B, respectively.

3.2 Parameter k

Parameter k is assigned by means of three bodies of evidence, each affected by both dissonance and imprecision. Dissonance and imprecision will be dealt with by discretizing each body of evidence by means of a random set. The problem of combining different bodies of evidence is an open one within Random Set Theory, and we will resort to Dempster's rule of combination, as will be justified below.

3.2.1 Dissonance and imprecision

Because the minimum, median and maximum values are assigned as compact intervals, for each source of information, there is an envelope of triangular distributions compatible with the given information.

For the *i*-th source of information, the CDFs belonging to such an envelope are bounded by an upper CDF, $U_{k,i}$, and a lower CDF, $L_{k,i}$. $U_{k,i}$ is defined by the triangular distribution in the interval $\left[\min_{Min_i}, \min_{Max_i}\right]$ with mode $\min_{Mod_i}$. $L_{k,i}$ is defined by the triangular distribution in the interval $\left[\max_{Min_i}, \max_{Max_i}\right]$ with mode $\max_{Mod_i}$. With the assigned numerical values, one obtains:

$$U_{k,1} = \begin{cases} \dfrac{(k-90)^2}{6600} & if \quad k \in [90,150] \\ \dfrac{-k^2 + 400 \cdot k - 34500}{5500} & if \quad k \in [150,200] \end{cases} \tag{7}$$

$$L_{k,1} = \begin{cases} \dfrac{(k-100)^2}{6600} & if \quad k \in [100,160] \\ \dfrac{-k^2 + 420 \cdot k - 38600}{5500} & if \quad k \in [160,210] \end{cases} \tag{8}$$

$$U_{k,2} = \begin{cases} \dfrac{(k-80)^2}{7200} & if \quad k \in [80,140] \\ \dfrac{-k^2 + 400 \cdot k - 32800}{7200} & if \quad k \in [140,200] \end{cases} \tag{9}$$

$$L_{k,2} = \begin{cases} \dfrac{(k-110)^2}{6600} & if \quad k \in [110,170] \\ -\dfrac{k^2}{5500} + \dfrac{2 \cdot x}{25} - \dfrac{39}{5} & if \quad k \in [170,220] \end{cases} \tag{10}$$

$$U_{k,3} = \begin{cases} \dfrac{(k-60)^2}{7800} & if \quad k \in [60,120] \\ \dfrac{-k^2 + 380 \cdot k - 27000}{9100} & if \quad k \in [120,190] \end{cases} \tag{11}$$

$$L_{k,3} = \begin{cases} \dfrac{(k-120)^2}{6600} & if \quad k \in [120,180] \\ \dfrac{-k^2 + 460 \cdot k - 47400}{5500} & if \quad k \in [180,230] \end{cases} \tag{12}$$

Because $U_{k,i}$, and $L_{k,i}$ may be generic functions and are not to satisfy the second equalities in Eqs. (A.6), in general, $U_{k,i}$, and $L_{k,i}$ are not Belief and Plausibility functions of a random set. However, since the system response must be treated as a black box, each parameter must be discretized in any case. Therefore, we propose to discretize each parameter in such a way that its upper and lower probabilities, $U_{k,i}$, and $L_{k,i}$, are Belief and Plausibility functions, respectively, of a random set ($\mathcal{F}_{k,i}$, $M_{k,i}$).

Two discretization methods are proposed:

- Averaging Discretization Method (ADM).
- Outer Discretization Method (ODM).

Consider Figures 7a and 7b. In the first step of both methods, the [0, 1] ordinate intervals of $U_{k,i}$ and $L_{k,i}$ are both discretized into n subintervals of length $M_j > 0$ ($j = 1,\ldots,n$); for example, $n = 5$ in Figures 7. By definition, let $M_0 := 0$, and let $U_{k,i}^{-1}$ and $L_{k,i}^{-1}$ indicate the inverse functions of $U_{k,i}$ and $L_{k,i}$, respectively.

According to the ADM, the j-th focal element of ($\mathcal{F}_{k,i}$, $M_{k,i}$) is the interval (see Figure 7a):

$$A_{k,i,j} = \left[U_{k,i}^{-1}\left(\sum_{s=0}^{j-1} M_s + \frac{M_j}{2} \right), L_{k,i}^{-1}\left(\sum_{s=0}^{j-1} M_s + \frac{M_j}{2} \right) \right] \qquad j = 1\ldots,n \tag{13.a}$$

and its basic probability assignment is:

$$M_{k,i}(A_{k,i,j}) = M_j \tag{13.b}$$

In the following, as a first approach to the problem in hand, subintervals of equal length 1/n are used. In Discretization A, n = 10, whereas in Discretization B, n = 20. Then, Eqs. (13.a) and (13.b) simplify as, respectively:

$$A_{k,i,j} = \left[U_{k,i}{}^{-1}\left(\frac{1}{n}\cdot\left(j-\frac{1}{2}\right)\right), L_{k,i}{}^{-1}\left(\frac{1}{n}\cdot\left(j-\frac{1}{2}\right)\right)\right] \qquad j = 1\ldots,n \tag{13.c}$$

$$M_{k,i}(A_{k,i,j}) = 1/n \tag{13.d}$$

According to the ODM, the j-th focal element of ($\mathcal{F}_{k,i}$, $M_{k,i}$) is the interval (see Figure 7b):

$$A_{k,i,j} = \left[U_{k,i}{}^{-1}\left(\sum_{s=0}^{j-1} M_s\right), L_{k,i}{}^{-1}\left(\sum_{s=1}^{j} M_s\right)\right] \qquad j = 1\ldots,n \tag{14.a}$$

where:

$$U_{k,i}{}^{-1}(0) := \lim_{x\to 0+} U_{k,i}{}^{-1}(x)$$

$$L_{k,i}{}^{-1}(1) := \lim_{x\to 1-} L_{k,i}{}^{-1}(x)$$

and its basic probability assignment is:

$$M_{k,i}(A_{k,i,j}) = M_j \tag{14.b}$$

If subintervals of equal length 1/n are used, then Eqs. (14.a) and (14.b) simplify as, respectively:

$$A_{k,i,j} = \left[U_{k,i}{}^{-1}\left(\frac{1}{n}\cdot(j-1)\right), L_{k,i}{}^{-1}\left(\frac{1}{n}\cdot(j)\right)\right], \; j = 1\ldots,n \tag{14.c}$$

$$M_{k,i}(A_{k,i,j}) = 1/n \tag{14.d}$$

ADM averages $U_{k,i}$ and $L_{k,i}$, and therefore it is very effective for calculating the expectation of a parameter or the expectation of its image through a function; however, as shown below, it may give a poor approximation to the tails.

On the other hand, the ODM ensures that $F_{upp}(k) \geq U_{k,i}(k)$ and that $F_{low}(k) \leq L_{k,i}(k)$. However, at present, it has not been proven that the same inclusions apply to the images of $(\mathscr{F}_{k,i}, M_{k,i})$ and of $U_{k,i}$ and $L_{k,i}$ through a generic function. Also, ODM may severely overestimate $U_{k,i}$ and understimate $L_{k,i}(k)$. Therefore, in the following, ADM will be used, and a deeper comparison between ADM and ODM will be the subject of a future paper.

Tables 2a and 2b give the focal elements for Discretizations A and B, respectively; in each table, columns 2, 3, and 4 are relevant to the first, second, and third source of information, respectively. Figures 8, 9, and 10 plot $U_{k,i}$ and $L_{k,i}$, along with F_{upp}, F_{low}, and the focal elements for the first, second, and third source of information, respectively. They confirm that, for the *i*-th source of information, F_{upp} and F_{low} calculated by means of Eqs. (A.7a-b) are a stepwise approximation to $U_{k,i}$ and $L_{k,i}$, respectively.

3.2.2 Combination of the three bodies of evidence

Within Random Set Theory, the combination of two or more bodies of evidence is still an open and unsolved problem. Within Evidence Theory, and, more broadly, within axiomatic approximate reasoning theories, several rules have been proposed (and criticized) in the literature and have been recently surveyed and contrasted by Sentz and Ferson (2002). This study confirmed that Dempster's rule of combination (Dempster, 1967) performs satisfactorily under situations of low conflict, in which each information source is equally reliable and all information sources are independent (more precisely, conditionally independent according to Walley (1991, page 276)) and consistent (Bernardini, 1999).

As already stated in Section 2, the problem statement (Oberkampf *et al.*, 2001) specifies that the three information sources are independent, equally credible, and consistent (see Eq. (3)). Therefore, the three random sets ($\mathcal{F}_{k,i}$, $M_{k,i}$), i = 1, 2, 3 are combined into a unique random set ($\mathcal{F}_k$, M_k) by extending to three random sets the Dempster's rule of combination found in textbooks (e.g. Klir and Yuan, 1995) as follows:

$$M_k(A) = \frac{\sum_{A=B\cap C\cap D} M_{k,1}(B)\cdot M_{k,2}(C)\cdot M_{k,3}(D)}{1-K} \tag{15}$$

for all $A \neq \emptyset$, and $M_k(A) \neq 0$, where:

$$K = \sum_{B\cap C\cap D=\emptyset} M_{k,1}(B)\cdot M_{k,2}(C)\cdot M_{k,3}(D) \tag{16}$$

According to Klir and Yuan (1995), the Dempster's rule of combination rests on the same ground used to justify the joint probability distribution of independent variables starting from their marginal distributions. However, since some intersections of focal elements from the three sources of information may result in the same set A, one must sum the corresponding products to obtain $M_k(A)$. Moreover, by dividing by the normalizing factor 1-K, one takes into account that some intersections of focal elements may be empty.

By using Discretization A (resp. B), the final random set ($\mathcal{F}_k$, M_k) contains 55 (resp. 216) focal elements; therefore, it was not possible to list all focal elements in tables. The Upper and the Lower CDFs relative to ($\mathcal{F}_k$, M_k) are portrayed in Figures 11a and 11b for Discretizations A and B, respectively. We notice that the Upper and the Lower CDFs in Figures 11a and 11b look closer together than the Upper and Lower CDFs (or $U_{k,i}$ and $L_{k,i}$) plotted in Figures 8, 9, and 10. In order to quantify this phenomenon, let us introduce the L_1 distance between two functions (e.g. Kolmogorov and Fomin, 1957):

$$d(f,g)=\int_{-\infty}^{\infty}\left|f(k)-g(k)\right|dk \tag{17}$$

When applied to Upper and Lower CDFs (or to $U_{k,i}$ and $L_{k,i}$) this distance is a measure of the imprecision affecting parameter *k*. One obtains:

$$d(U_{k,1},L_{k,1})=10 \tag{18}$$

$$d(U_{k,2},L_{k,2})=26.67 \tag{19}$$

$$d(U_{k,3},L_{k,3})=53.33 \tag{20}$$

For the final random set ($\mathcal{F}_k$, M_k), distances take on the following values:

- Discretization A: $d(F_{upp},F_{low})=6.68249$ (21)

- Discretization B: $d(F_{upp},F_{low})=6.73472$ (22)

The combination of the three independent bodies of information into ($\mathcal{F}_k$, M_k) resulted in a less imprecise information than the starting bodies of information. This is because the Dempster's rule of combination (Eqs. 15 and 16) retains as focal elements only the non-empty intersections among the focal elements of the three independent sources of information. As an extreme case, if the focal elements of the three sources of information had no common element (i.e. if there were no consensus among the three sources of information), ($\mathcal{F}_k$, M_k) would contain no focal element, i.e. no information on *k* consistent with the three sources of information would be available.

From a computational viewpoint, we notice that the finer the discretization is, the larger the distance between F_{upp} and F_{low} is. For example, distance *d* increased by 7.8% when Discretization B was used rather than Discretization A; at the same time, the number of focal elements increased by 293%. As discussed in Section 4.2 below, this behavior is attributable to the coarse approximation to the CDFs tails operated by Discretization A.

3.3 Parameter c

Parameter c is affected by both dissonance and imprecision. The available information conforms exactly to the definition of a random set as given in the Appendix, with focal elements:

$$A_{c,1} = C_1 = [5, 10] \tag{23}$$

$$A_{c,2} = C_2 = [15, 20] \tag{24}$$

$$A_{c,3} = C_3 = [25, 25] \tag{25}$$

and with probability assignment:

$$M_c(A_{c,i}) = 1/3, \quad i = 1, 2, 3 \tag{26}$$

3.4 Parameter ω

Parameter w is affected by both dissonance and imprecision. By following the same procedure as in Section 3.2.1, one obtains:

$$U_w = \begin{cases} 2 \cdot (w-2)^2 & \text{if} \quad w \in [2, 2.5] \\ -2 \cdot (w - 3.7071) \cdot (w - 2.2929) & \text{if} \quad w \in [2.5, 3] \end{cases} \tag{27}$$

$$L_w = \begin{cases} 2.0833 \cdot (w-2.3)^2 & \text{if} \quad w \in [2.3, 2.7] \\ -1.04167 \cdot (w - 4.4798) \cdot (w - 2.502) & \text{if} \quad w \in [2.7, 3.5] \end{cases} \tag{28}$$

Random sets are constructed by using Eqs. (13c) and (13d), and by discretizing the [0, 1] ordinate intervals of U_ω and L_ω by means of 10 intervals (Discretization A) and 20 intervals (Discretization B).

The results are given in Tables 3a and 3b for Discretizations A and B, respectively, whereas Figures 12a and 12b illustrate the Upper and the Lower CDFs obtained with Discretizations A and B, respectively.

3.5 Random relation constraining all parameters

Each parameter is now constrained by a random set. In the problem statement, the parameters are stated to be independent. In Probability Theory, the joint probability distribution of *n* independent random variables ($x_1,\dots, x_n$) can be calculated starting from the marginal distributions as follows:

$$p(x_1,\dots,x_n) = p(x_1)\cdot\dots\cdot p(x_n) \tag{29}$$

Eq. (29) is straightforwardly extended to variables constrained by random sets and the resulting random relation is called *stochastically decomposable random Cartesian product* (Dubois and Prade, 1991). Let $M_i(A_i)$ be the basic probability assignment of the *i*-th parameter. Then the focal elements of the stochastically decomposable random relation ($\mathcal{F}$, M) are all Cartesian products $A = A_1\times,\dots,\times A_n$, and the joint basic probability assignment is defined as:

$$M(A_1\times,\dots,\times A_n) = M_1(A_1)\cdot\dots\cdot M_n(A_n) \tag{30}$$

More explicitly, in our case:

$$\mathcal{F} = \{A = A_{m,i}\times A_{k,j}\times A_{c,h}\times A_{w,l},\ for\ all\ combinations\ of\ i, j, h, l\} \tag{31.a}$$

$$M(A = A_{m,i}\times A_{k,j}\times A_{c,h}\times A_{w,l}) = M_m(A_{m,i})\cdot M_k(A_{k,j})\cdot M_c(A_{c,h})\cdot M_w(A_{w,l}) \tag{31.b}$$

4. Extending the parameter uncertainty through function D_s

The extension principle for random sets (Eqs. A.8a and A.8b) is used to map the random relation ($\mathcal{F}$, M) defined in Eqs. (31.a) and (31.b) to the mechanical system response through function D_s (Eq. 4). This extension principle is the straightforward extension to the range of a function used in Set Theory (Dubois and Prade, 1991) and, when ($\mathcal{F}$, M) is a stochastically decomposable random Cartesian product, it specializes in:

$$\mathcal{R} = \left\{ R = D_s\left(A_{m,i}, A_{k,j}, A_{c,h}, A_{w,l}\right) \right\} \tag{31.c}$$

$$r(R) = \left\{ \sum M\left(A_{m,i}\right) \cdot M\left(A_{k,j}\right) \cdot M\left(A_{c,h}\right) \cdot M\left(A_{w,l}\right) \mid R = D_s\left(A_{m,i}, A_{k,j}, A_{c,h}, A_{w,l}\right) \right\} \tag{31.d}$$

As pointed out by Dubois and Prade (1991) this is the definition of a function of stochastically independent random-set valued arguments, as first suggested by Yager (1986).

4.1 Computational aspects

4.1.1 Calculation of the image of a focal element

Eq. (31.c) requires the calculation of the image of a focal element A_i through function D_s. This entails solving the two global optimization problems:

$$D_s(A_i) = [l_i, r_i] \quad \text{where} \tag{32.a}$$

$$l_i = \min_{u \in A_i} D_s(u) \tag{32.b}$$

$$r_i = \max_{u \in A_i} D_s(u) \tag{32.c}$$

Because A_i is the Cartesian product of 4 intervals, it is a 4-dimensional box with 2^4 vertices. Tonon *et al.* (2000b) discuss methods for solving problems (32.b) and (32.c) when monotonicity properties of function D_s are known.

If monotonicity properties of function D_s are not known, then it is proposed to calculate function D_s at the 16 vertices $v_{i,j}$ of the 4-dimensional box A_i, and to assume:

$$l_i = \min\{D_s(v_{i,j}), j = 1,\ldots,16\} \qquad (33.a)$$

$$r_i = \max\{D_s(v_{i,j}), j = 1,\ldots,16\} \qquad (33.b)$$

If D_s has no extreme point in the interior of A_i or on its edges, then Eqs. (33.a) and (33.b) are correct, and are simply the application of the vertex method (Dong and Shah, 1987; Dong and Wang, 1987; Dong *et al.* 1987). If, on the other hand, D_s has one or more extreme points in the interior of A_i or on its edges, then Eqs. (33.a) and (33.b) can be taken as approximations to the true global extreme values; as the discretizations introduced in Section 3 become finer, the accuracy of these approximations increases.

It is the author's opinion that, from a computational viewpoint, it is more efficient to increase the fineness of the parameter discretizations introduced in Section 3 than to invoke a global optimization tool at each focal element image calculation. Indeed, by increasing the fineness of the parameter discretizations, one achieves three objectives:

1) A better approximation to the assigned parameter distributions because the number of focal elements of each parameter increases.
2) A finer granularity of the CDFs of D_s, see also Section 4.1.2.
3) A better approximation to the global maxima and minima of function D_s because focal elements A_i become smaller.

For example, in the parameter ranges considered here, function D_s is an increasing function of m, a decreasing function of c, an increasing function of w, but it is not a monotonic function of k (resonance). Therefore, the maximum value of D_s is achieved on an edge of A_i.

For Discretization A, the exact maximum value of D_s in $\bigcup A_i$ is 7.937404847026513, and the maximum value calculated by means of Eq. (33.b) in $\bigcup A_i$ is 7.936812991685837, with a relative difference of $7.4 \cdot 10^{-5}$.

For Discretization B, the exact maximum value of D_s in $\bigcup A_i$ is 8.090234385162086, and the maximum value calculated by means of Eq. (33.b) in $\bigcup A_i$ is 8.090207265232385, with a relative difference of $3.3 \cdot 10^{-6}$.

4.1.2 Granularity of the CDFs of function D_s

Once the image $(\mathcal{R}, r)$ of $(\mathcal{F}, M)$ through function D_s has been calculated by means of the extension principle, it is possible to calculate the Upper and the Lower CDFs of D_s by means of Eqs. (A.7a) and (A.7b).

Let n_m, n_k, n_c, n_ω be the number of focal elements for parameters m, k, c, and w, respectively. Unless some focal elements A_i have the same image through function D_s, the granularity of the CDFs of function D_s is in the order of $1/(n_m \cdot n_k \cdot n_c \cdot n_\omega)$.

For example, for Discretization A, the granularity is in the order of 10^{-5}, whereas for Discretization B, the granularity is in the order of 10^{-6}.

If the CDFs of a particular value of D_s, say D_s*, are the only results of interest, as opposed to the complete CDFs, then a special technique can be used (Tonon *et al.*, 2000b), which can be summarized as follows.

At first, a coarse parameter discretization is used. The focal elements A_i whose r_i in Eq. (32.a) is less than D_s* need not be further discretized because their discretizations will always map to the left of D_s* on the real line, so that their contribution to the CDFs of D_s* will not change (see Eqs. A.7a and A.7b). Likewise, the focal elements A_i whose l_i in Eq. (32.a) is greater than D_s* need not be further discretized because their discretizations will always map to the right of D_s* on the real line, so that they do not contribute to the CDFs of D_s* altogether (see Eqs. A.7a and A.7b). The only focal elements A_i that need to be further discretized are those such that $D_s^* \in [l_i, r_i]$. As shown in (Tonon *et al.*, 2000b), this procedure leads to a computational savings of orders of magnitudes in the number of function calls as compared to the calculation of the complete CDFs.

4.2 Results

The imprecision affecting parameters k, c, and w is mapped through function D_s, and consequently the CDF of D_s is not unique, but is bounded between an upper and a lower bound (Eqs. (A.7a) and (A.7b)). Figures 13a and 13b show the CDFs of the dependent variable D_s for Discretizations A and B, respectively. The expectation of D_s for Discretizations A and B can be calculated by means of Eq. (A.7c):

- Discretization A: $m_A = [1.58618, 2.13159]$
- Discretization B: $m_B = [1.59059, 2.12184]$

and their relative differences are as follows:

$$(\min m_B - \min m_A)/\min m_A = 0.28\% \qquad (34.a)$$

$$(\max m_B - \max m_A)/\max m_A = -0.46\% \qquad (34.b)$$

Although the number of D_s function calls used in Discretization B is 15.71 times the respective number used in Discretization A, the relative differences between minimum and maximum values of the expectations are less than 0.5%. This is confirmed by the fact that, if plotted on the same graph, the Upper (resp. Lower) CDF for Discretization A is indistinguishable from the Upper (resp. Lower) CDF for Discretization B.

For verification purposes, Tables 4a and 4b give values of the Upper and the Lower CDFs, respectively, calculated at selected abscissas by means of Discretizations A and B. Columns 4 in Tables 4a and 4b present the relative differences between the CDFs calculated by means of Discretizations A and B, respectively. As for the Upper CDF (Table 4a), the relative difference is always positive, and diminishes as D_s increases. When the Lower CDF is considered (Table 4b), the relative difference diminishes and changes sign as D_s increases: it is positive for low values of D_s, and negative for larger values of D_s.

These trends are attributable to the uniform step used in the discretization of the three sources of information on k (Section 3.2). In fact, consider Figures 8, 9, and 10. The lower tails of the CDFs of k grow slowly, and therefore the first focal element of Discretization A is only a rough approximation to the lower tails of the CDFs, and it misses the lower extreme values of k. The central parts of the CDFs of k grow fast, and the approximations by means of focal elements are more accurate. Therefore, because the CDFs of D_s are cumulative functions, the effect of the lower extreme values of k is less significant for larger values of D_s. Likewise, the upper tails of the CDFs of k grow slowly, and thus the tenth focal element of Discretization A is only a rough approximation to the upper tails of the CDFs, and it misses the extreme values of k. This causes the change in sign in the relative difference between the Lower CDFs of D_s in Table 4b.

5. Conclusions and future work

In the assigned problem formulation, the parameters controlling the response of a mechanical system were assumed to be affected by both dissonance and imprecision. In the procedure proposed herein, both types of uncertainty were propagated to the mechanical system response by means of Random Set Theory. As a result, Upper and Lower Cumulative Distribution Functions could be calculated for the mechanical system response, that are an envelope to all CDFs compatible with the available information on the parameters.

In order for the system response to be treated as a “black box”, different methods were introduced for discretizing the parameters. Each method was tailored to the type of information available on a specific parameter. In any case considered, the result of each discretization method was a random set.

When the system response presents maxima and minima in the range of parameters considered, the computational experience gained so far indicated that it is better to increase the fineness of the parameter discretization than to invoke a global optimization tool.

The results obtained using two discretizations (a coarser one and a finer one) indicated that the expectation of the response differed by less than 0.5% when the number of function calls was increased 15.6 times.

However, larger differences (4-5%) were obtained for the lower tails of the CDFs of the response. Further research is necessary in order to investigate the benefits of a non-uniform discretization step for the parameters aimed at increasing the accuracy of the lower tails.

When generic upper and lower probabilities are assigned to a parameter, two random-set approximation methods have been proposed, namely the Averaging Discretization Method (ADM) and the Outer Discretization Method (ODM). The relative merits of ADM versus ODM need to be further studied; the inclusion properties of ODM and of finer ODM discretizations must be investigated.

In the assigned problem formulation, parameters and sources of information were assumed to be independent. Further work is necessary in order to advance our understanding and to investigate the computational aspects of correlation among the parameters.

6. Acknowledgement

The author would like to express his gratitude to Prof. A. Bernardini (Department of Construction and Transportation, University of Padova, Italy) for introducing him to Random Set Theory and for his precious comments on the first draft of this manuscript.

7. References

Bernardini A. What are the random and fuzzy sets and how to use them for uncertainty modelling in engineering systems? In *Whys and Hows in Uncertainty Analysis: Probability, Fuzziness and Anti-Optimization* (Elishakoff I., ed.), pp. 63-125. Springer, Vienna (1999).

Dempster A. P. Upper and lower probabilities induced by a multivalued mapping. *Ann. Math. Stat.*, Vol. 38, pp.325-339 (1967).

Dong W. M. and Shah H.C. Vertex method for computing functions of fuzzy variables. *Fuzzy Sets and Systems*, Vol. 24, pp. 65-78 (1987).

Dong W. M. and Wong F.S. Fuzzy Weighted averages and implementation of the extension principle. *Fuzzy Sets and Systems* Vol. 21, pp. 183-199 (1987).

Dong W. M., Chiang W. L. and Shah H.C. Fuzzy information processing in seismic hazard analysis and decision making. *Soil Dynamics and Earthquake Engineering* Vol. 6, No. 4, 220-226 (1987).

Dubois D. and Prade H. Random sets and fuzzy interval analysis. *Fuzzy Sets and Systems*, Vol. 42, pp. 87-101 (1991).

Klir G. J. and Yuan B. *Fuzzy Sets and Fuzzy Logic, Theory and Application*, Prentice Hall, Upper Saddle River, New Jersey (1995).

Klir, G. J. Is there more to uncertainty than some probability theorists might have us believe?. *Int. J. General Systems*, Vol. 15, pp. 347-378 (1989).

Kolmogorov, A.N. and Fomin S.V. *Elements of the Theory of Functions and Functional Analysis*, Graylock Press, Rochester, New York (1957).

Oberkampf W. L., Helton J. C., Joslyn C. A., Wojtkiewicz S. F., and Ferson S. Uncertainty in system response given uncertain parameters. http://www.sandia.gov/epistemic/ (2001).

Sentz, K. and Ferson, S. *Combination of Evidence in Dempster-Shafer Theory*. Sandia National Laboratories, SAND2002-0835 (2002).

Tonon F., Bernardini A. and Mammino A. Determination of parameters range in Rock Engineering by means of Random Set Theory. Reliability Engineering & System Safety, Vol. 70(3), pp. 241-261 (2000a).

Tonon F., Bernardini A. and Mammino A. Reliability analysis of rock mass response by means of Random Set Theory. Reliability Engineering & System Safety, Vol. 70(3), pp.263-282 (2000b).

Yager, R.R. (1986). Arithmetic and other operations on Dempster-Shafer structures. *Int. J. Man-Machine Studies*, Vol. 25, pp. 357-366 (1986).

Walley P. *Statistical Reasoning with Imprecise Probabilities.* Chapman and Hall, London (1991).

8. Appendix

Suppose N observations were made of a parameter $u \in U$, each of which resulted in an imprecise (non-specific) measurement given by a set A of values. Let n_i denote the number of occurrences of the set $A_i \subseteq U$, and let $\mathcal{P}(U)$ denote the set of all the subsets of U (power set of U). A frequency function M can be defined, called *basic probability assignment*, such that:

$$M: \mathcal{P}(U) \rightarrow [0,1] \; ; \tag{A.1}$$

$$M(\varnothing) = 0 \; ; \tag{A.2}$$

$$\sum_{A \in \mathcal{P}(U)} M(A) = 1 \; ; \tag{A.3}$$

(In the literature, a lower-case m is often used to indicate the basic probability assignment; however, a lower-case m is here used to indicate the mass of the mechanical system).

According to Dempster (1967), this function can be obtained as follows. Consider a probability measure $P(z)$ defined on a universal set Z (which can be thought of as the set of our observations) related to U (the set of the values of our measurements) through a multivalued (one-to-many) mapping $\Gamma : Z \rightarrow \mathcal{P}(U)$. Then the basic probability assignment is:

$$M(A_i) = P(z_i) = n_i / N \tag{A.4}$$

This multivalued mapping expresses the imprecision of the measurement experienced during each observation, i.e. our inability to attach a single number to each observation. So, for each set $A \in \mathcal{P}(U)$, the value $M(A_i)$ expresses the probability of $z_i = \Gamma^{-1}(A_i)$ ($z_i \in Z$) and it does not exclude that the subsets of A_i can get additional probability deriving from other subsets B of U such that $A_i \cap B \neq \varnothing$. If $M(A_i) > 0$

(i.e. if A_i has occurred at least once), A_i is said *focal element*. A *random set* is the pair ($\mathcal{F}$, M) where $\mathcal{F}$ is the family of all focal elements. If u is a vector of two or more parameters, ($\mathcal{F}$, M) is called *random relation*.

Because of the presence of imprecision, it is not possible to calculate the probability of a generic $u \in U$ or of a generic subset $E \subseteq U$, but only a lower and an upper bound on this probability:

$$Bel(E) \leq Pro\,(E) \leq Pl(E) \tag{A.5}$$

where, denoted E^c the complement of E:

$$Bel(E) = \sum_{A_i : A_i \subseteq E} m(A_i) = 1 - Pl(E^c) \tag{A.6a}$$

$$Pl(E) = \sum_{A_i : A_i \cap E \neq \varnothing} m(A_i) = 1 - Bel(E^c) \tag{A.6b}$$

$Bel(E)$ is called *belief measure* and is the sum of the frequencies of those focal elements contained in E and whose occurrence *must* then lead to the claim that $u \in E$. $Pl(E)$ is said *plausibility measure* and is the sum of the frequencies of those focal elements having some element u in common with E and whose occurrence *may* lead to the claim that $u \in E$.

When U is the real line, the two limit cases:

(i) every weight $M(A)$ is concentrated on the lower bound of the focal element A;

(ii) every weight $M(A)$ is concentrated on the upper bound of the focal element A,

lead to two limit cumulative probability distribution functions (Bernardini, 1999):

$$F_{upp}(u) = Pl(\{u' \in U : u' \leq u\}) = \sum_{A_i : u \geq \inf(A_i)} M(A_i) \tag{A.7a}$$

$$F_{low}(u) = Bel(\{u' \in U : u' \leq u\}) = \sum_{A_i : u \geq \sup(A_i)} M(A_i) \tag{A.7b}$$

Eqs. (A.7a) and (A.7b) indicate that the calculation of the expectation m makes sense and is the interval (Dempster, 1967):

$$m = \left[\sum_{i=1}^{N} m_i \cdot \inf(A_i), \sum_{i=1}^{N} m_i \cdot \sup(A_i) \right] \qquad \text{(A.7c)}$$

Let $y=f(u)$, $f: U \rightarrow Y$ be a function of u. One is interested in getting information on the random set $(\mathscr{R}, r)$, which is the image of $(\mathscr{F}, M)$ through f. This is accomplished by the following extension principle (Dubois and Prade, 1991):

$$\mathscr{R} = \left\{ R_i = f(A_j) : A_j \in \mathscr{F} \right\} \qquad \text{(A.8a)}$$

$$r(R_i) = \sum_{A_j : R_i = f(A_j)} M(A_j). \qquad \text{(A.8b)}$$

FIGURES

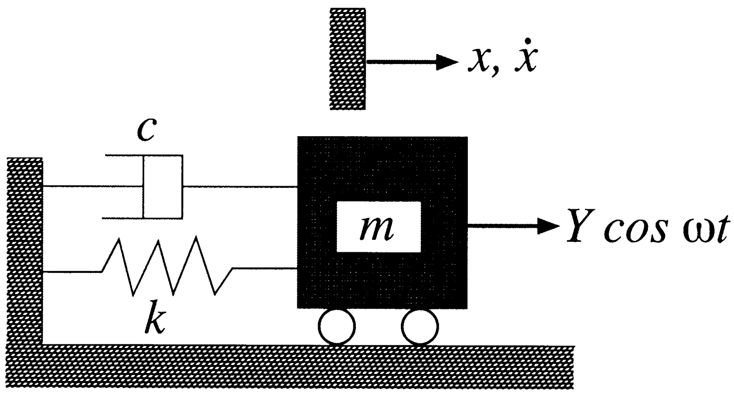


Figure 1. Mass-spring-damper system acted on by an excitation function.

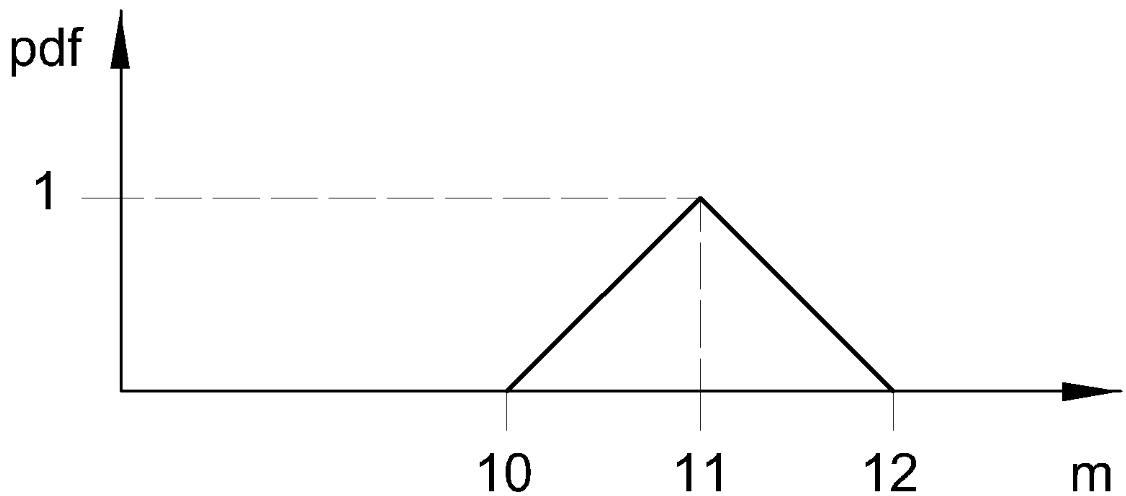


Figure 2. Probability distribution function of parameter *m*.

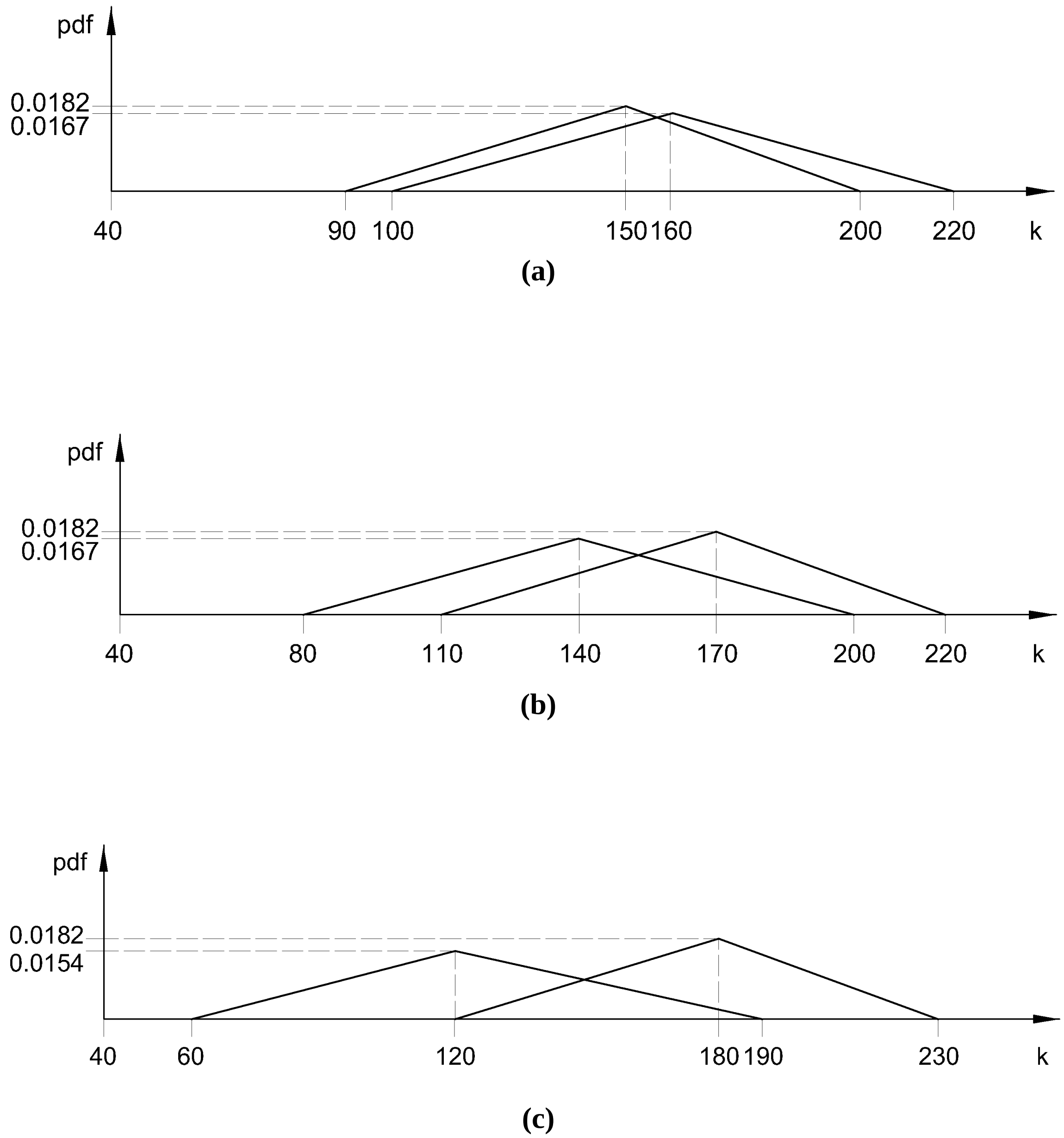


Figure 3. Probability distribution functions of parameter *k*: a) first source of information, b) second source of information, c) third source of information.

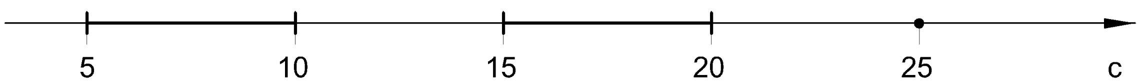


Figure 4. Intervals for parameter *c*.

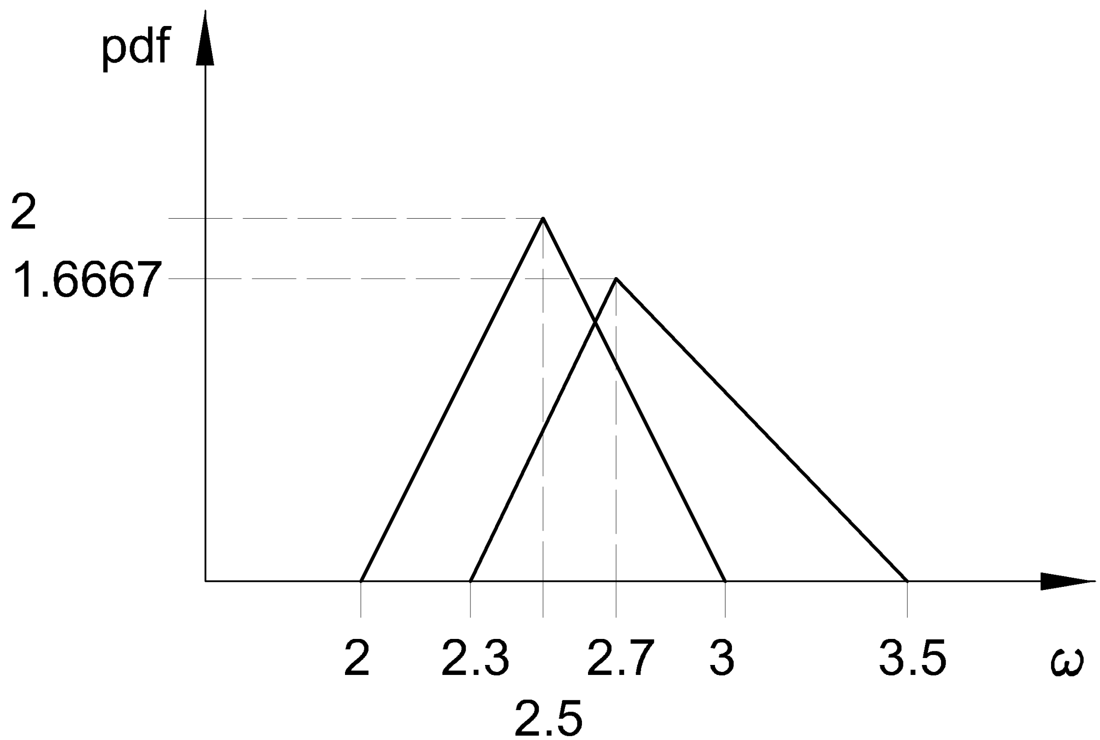
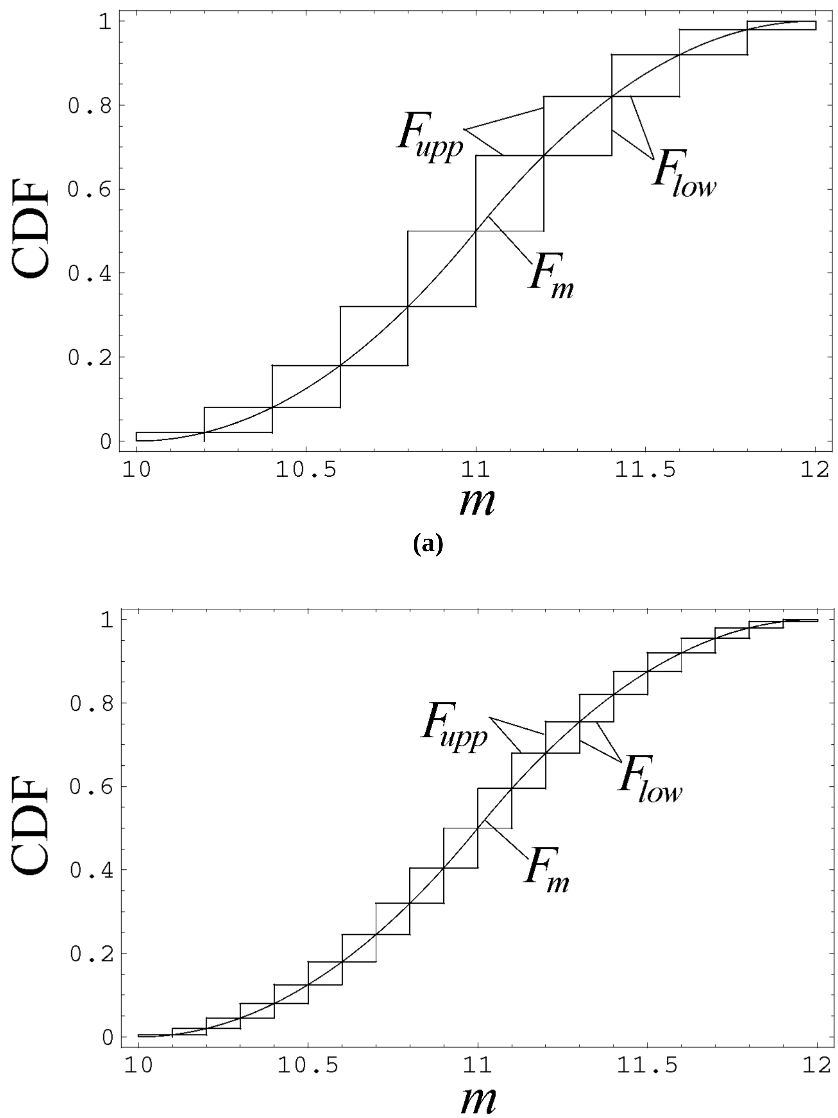


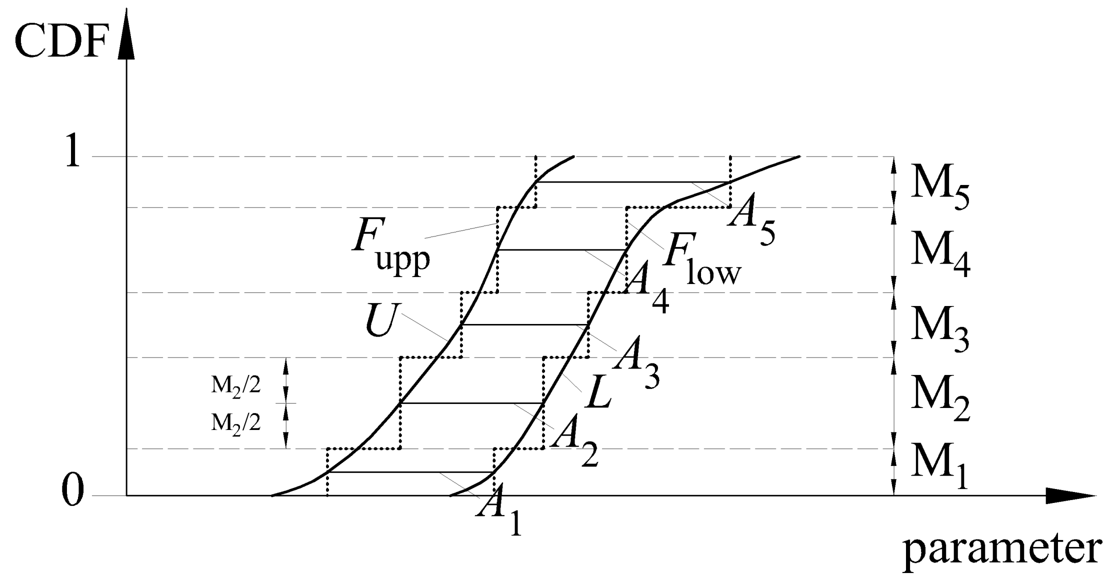
Figure 5. Probability distribution functions of parameter *w*.



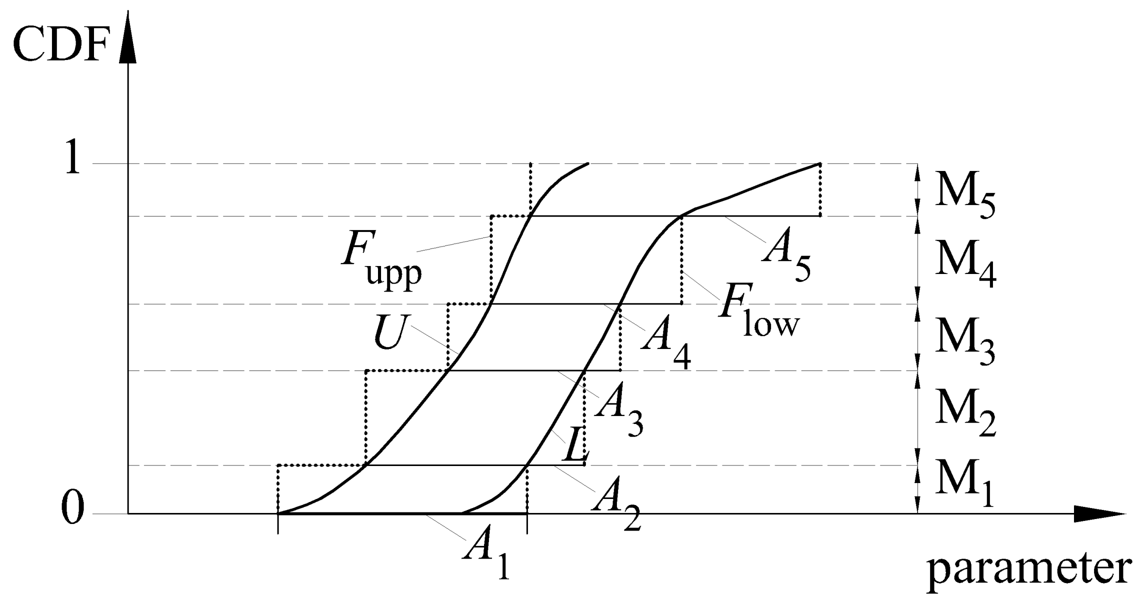
(a)

(b)

Figure 6. Cumulative Distribution Function (F_m) and Upper (F_{upp}) and Lower (F_{low}) CDFs of parameter *m*: a) Upper and Lower CDFs obtained with Discretization A, b) Upper and Lower CDFs obtained with Discretization B.

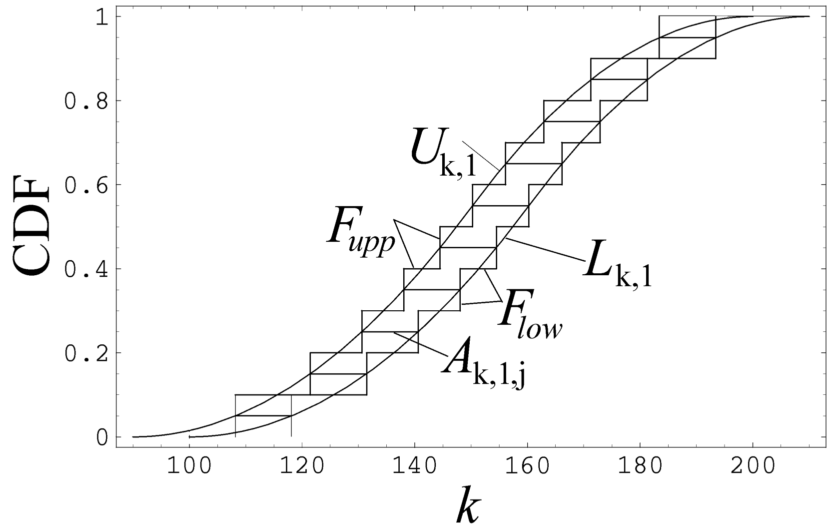


(a)

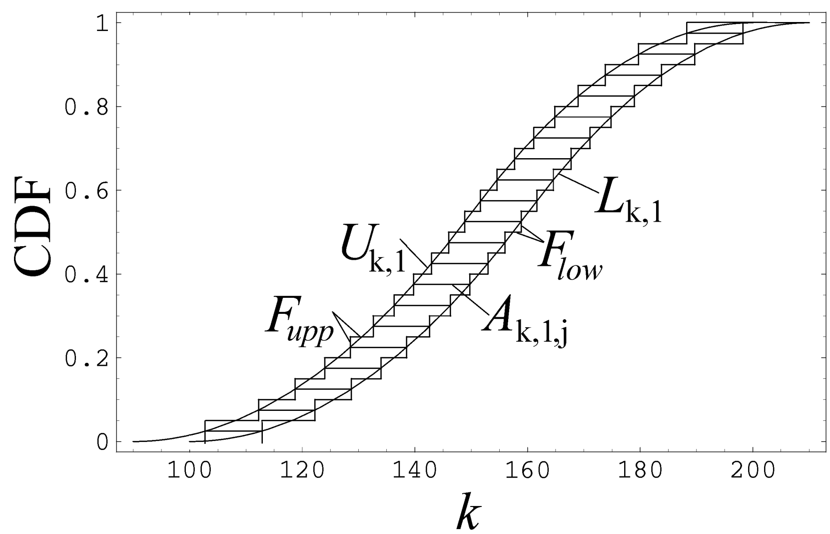


(b)

Figure 7. Discretization of CDFs *U* and *L* by means of a random set with 5 focal elements A_i. The Upper (F_{upp}) and Lower (F_{low}) CDFs of the random set are also shown. a) Averaging Discretization Method (ADM); b) Outer Discretization Method (ODM).

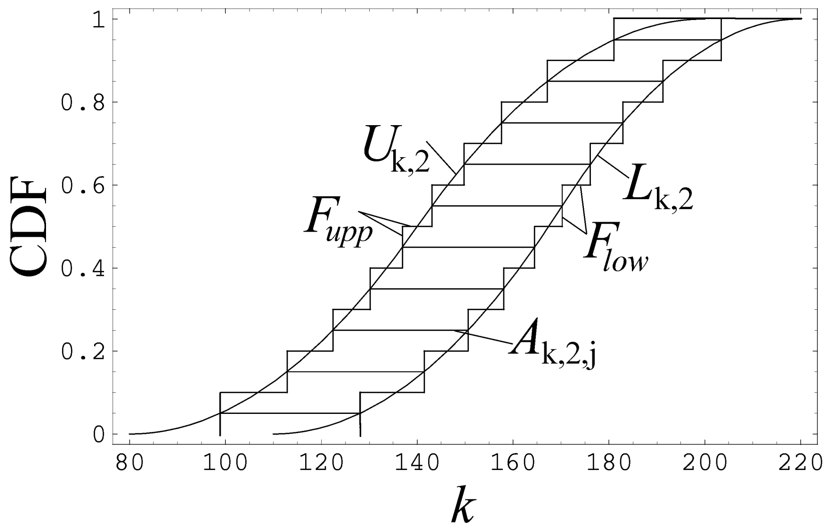


(a)

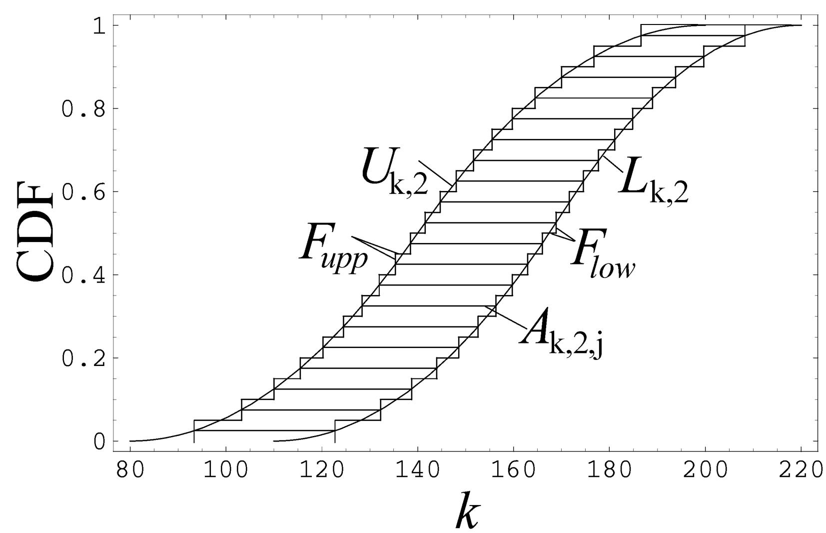


(b)

Figure 8. Upper ($U_{k,1}$) and Lower ($L_{k,1}$) CDFs of parameter k according to the first source of information: a) focal elements $A_{k,1,j}$ for Discretization A and Discretization A of the Upper CDF (F_{upp}) and of the Lower CDF (F_{low}); b) focal elements $A_{k,1,j}$ for Discretization B, Discretization B of the Upper CDF (F_{upp}) and of the Lower CDF (F_{low}).

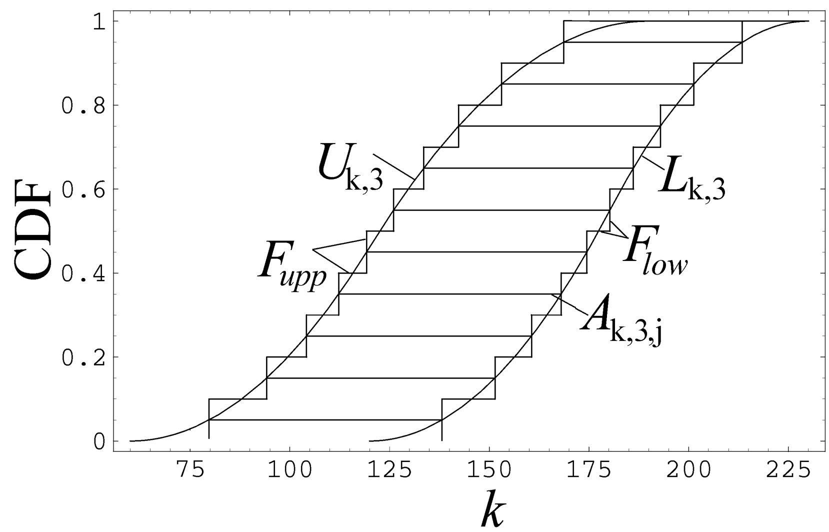


(a)

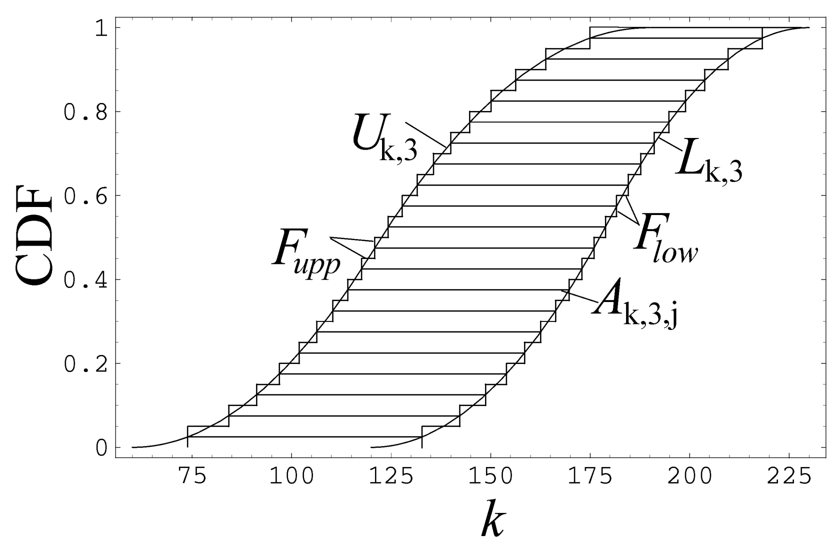


(b)

Figure 9. Upper ($U_{k,2}$) and Lower ($L_{k,2}$) CDFs of parameter *k* according to the second source of information: a) focal elements $A_{k,2,j}$ for Discretization A and Discretization A of the Upper CDF (F_{upp}) and of the Lower CDF (F_{low}); b) focal elements $A_{k,2,j}$ for Discretization B, Discretization B of the Upper CDF (F_{upp}) and of the Lower CDF (F_{low}).

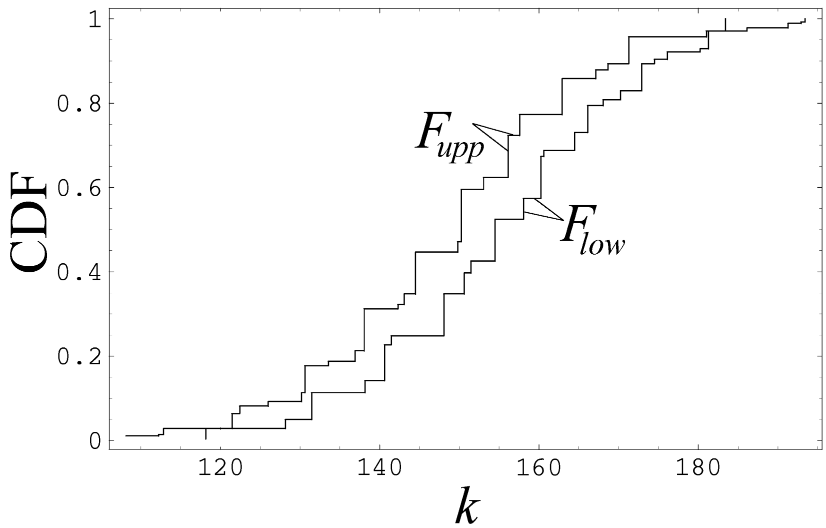


(a)

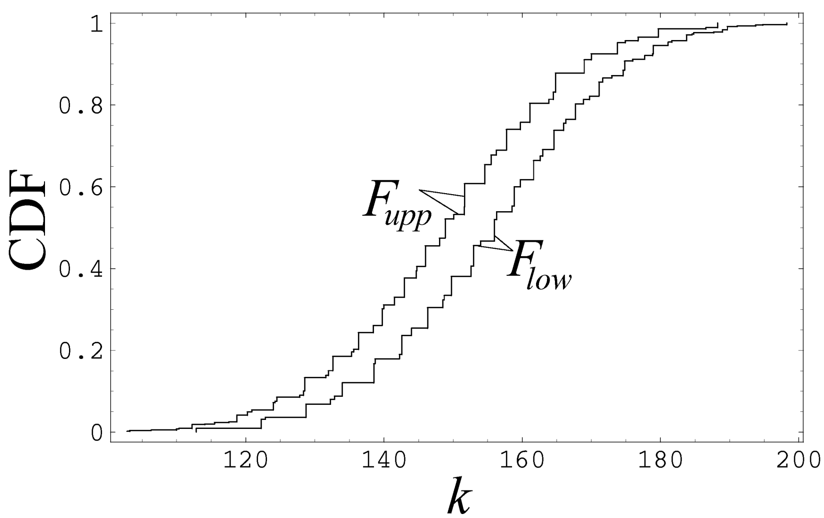


(b)

Figure 10. Upper ($U_{k,3}$) and Lower ($L_{k,3}$) CDFs of parameter k according to the third source of information: a) focal elements $A_{k,3,j}$ for Discretization A and Discretization A of the Upper CDF (F_{upp}) and of the Lower CDF (F_{low}); b) focal elements $A_{k,3,j}$ for Discretization B, Discretization B of the Upper CDF (F_{upp}) and of the Lower CDF (F_{low}).



(a)



(b)

Figure 11. Upper (F_{upp}) and Lower (F_{low}) CDFs of parameter *k* obtained by combining the information from the three sources of information; a) Discretization A; b) Discretization B.

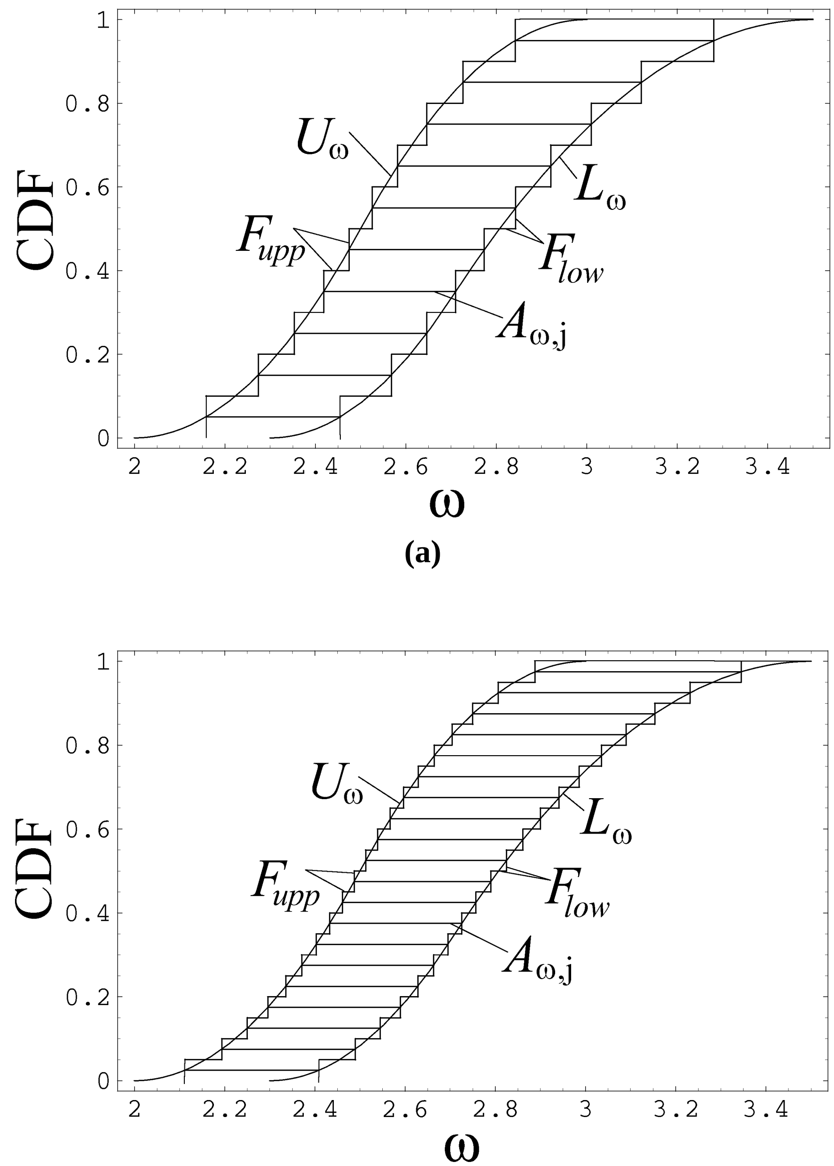
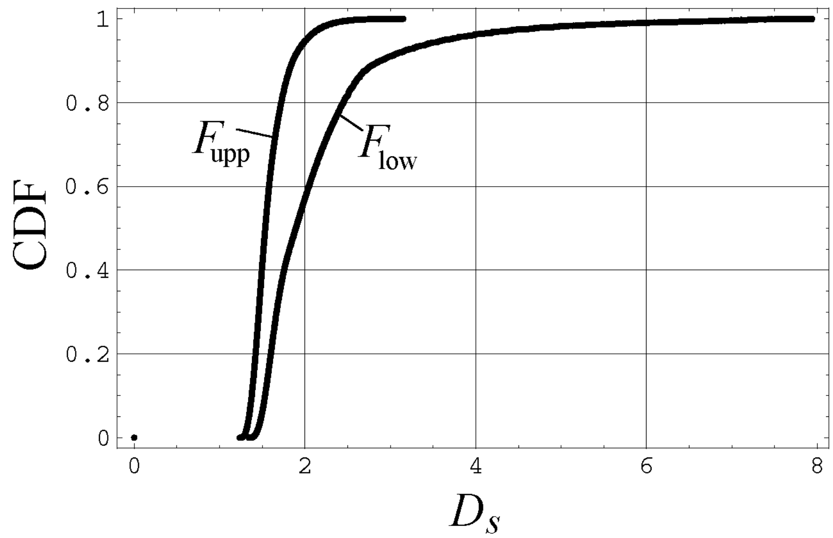
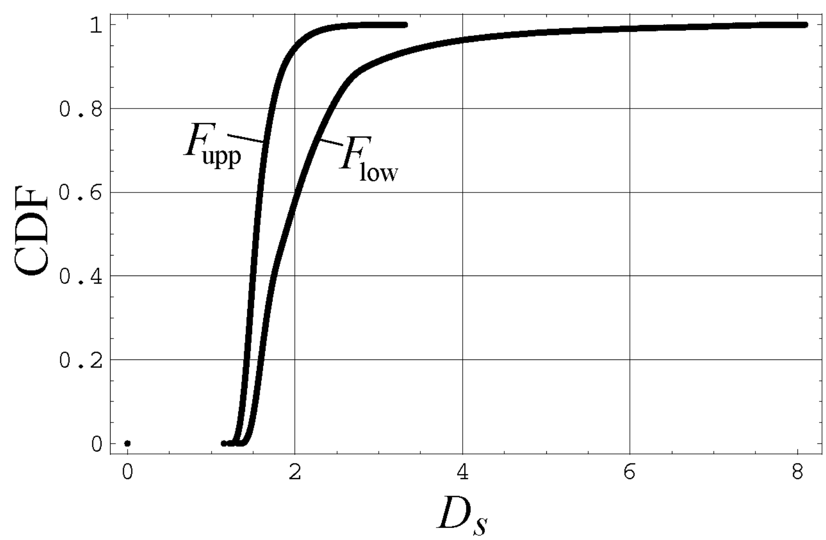


Figure 12. Upper (U_ω) and Lower (L_ω) CDFs of parameter *w*: a) focal elements $A_{\omega,j}$ for Discretization A and Discretization A of the Upper CDF (F_{upp}) and of the Lower CDF (F_{low}); b) focal elements $A_{\omega,j}$ for Discretization B, Discretization B of the Upper CDF (F_{upp}) and of the Lower CDF (F_{low}).



(a)



(b)

Figure 13. Upper (F_{upp}) and Lower (F_{low}) CDFs of the mechanical system response D_s: a) Discretization A; b) Discretization B.

TABLES

Table 1a. Discretization A of parameter *m* into 5 + 5 = 10 focal elements.

Focal element $A_{m,i}$	Basic probability assignment $M_m(A_{m,i})$
[10.0, 10.2]	0.02
[10.2, 10.4]	0.06
[10.4, 10.6]	0.1
[10.6, 10.8]	0.14
[10.8, 11.0]	0.18
[11.0, 11.2]	0.18
[11.2, 11.4]	0.14
[11.4, 11.6]	0.1
[11.6, 11.8]	0.06
[11.8, 12.0]	0.02

Table 1b. Discretization B of parameter *m* into 10 + 10 = 20 focal elements.

Focal element $A_{m,i}$	Basic probability assignment $M_m(A_{m,i})$
[10.0, 10.1]	0.005
[10.1, 10.2]	0.015
[10.2, 10.3]	0.025
[10.3, 10.4]	0.035
[10.4, 10.5]	0.045
[10.5, 10.6]	0.055
[10.6, 10.7]	0.065
[10.7, 10.8]	0.075
[10.8, 10.9]	0.085
[10.9, 11.0]	0.095
[11.0, 11.1]	0.095
[11.1, 11.2]	0.085
[11.2, 11.3]	0.075
[11.3, 11.4]	0.065
[11.4, 11.5]	0.055
[11.5, 11.6]	0.045
[11.6, 11.7]	0.035
[11.7, 11.8]	0.025
[11.8, 11.9]	0.015
[11.9, 12.0]	0.005

Table 2a. Focal elements of parameter k for Discretization A.

j	Focal element $A_{k,1,j}$	Focal element $A_{k,2,j}$	Focal element $A_{k,3,j}$
1	[108.1659, 118.1659]	[98.9737, 128.1659]	[79.7484, 138.1659]
2	[121.4643, 131.4643]	[112.8633, 141.4642]	[94.2053, 151.4643]
3	[130.6202, 140.6202]	[122.4264, 150.6202]	[104.1588, 160.6202]
4	[138.0624, 148.0624]	[130.1996, 158.0624]	[112.2494, 168.0624]
5	[144.4977, 154.4977]	[136.9210, 164.4977]	[119.2452, 174.4977]
6	[150.2506, 160.2506]	[143.0790, 170.2506]	[126.0078, 180.2506]
7	[156.1252, 166.1252]	[149.8004, 176.1252]	[133.5642, 186.1252]
8	[162.9190, 172.9190]	[157.5736, 182.9190]	[142.3030, 192.9190]
9	[171.2772, 181.2772]	[167.1366, 191.2772]	[153.0541, 201.2772]
10	[183.4169, 193.4169]	[181.0263, 203.4169]	[168.6693, 213.4169]

Table 2b. Focal elements of parameter k for Discretization B.

j	Focal element $A_{k,1,j}$	Focal element $A_{k,2,j}$	Focal element $A_{k,3,j}$
1	[102.8452, 112.8452]	[93.4164, 122.8452]	[73.9642, 132.8452]
2	[112.2486, 122.2486]	[103.2379, 132.2486]	[84.1868, 142.2486]
3	[118.7228, 128.7228]	[110.0000, 138.7228]	[91.2250, 148.7228]
4	[123.9853, 133.9853]	[115.4965, 143.9853]	[96.9458, 153.9853]
5	[128.5357, 138.5357]	[120.2492, 148.5357]	[101.8927, 158.5357]
6	[132.6028, 142.6028]	[124.4972, 152.6028]	[106.3141, 162.6028]
7	[136.3141, 146.3141]	[128.3735, 156.3141]	[110.3487, 166.3141]
8	[139.7493, 149.7493]	[131.9615, 159.7493]	[114.0833, 169.7493]
9	[142.9622, 152.9622]	[135.3173, 162.9622]	[117.5760, 172.9622]
10	[145.9911, 155.9911]	[138.4808, 165.9911]	[120.8806, 175.9911]
11	[148.8642, 158.8642]	[141.5192, 168.8642]	[124.2543, 178.8642]
12	[151.6523, 161.6523]	[144.6827, 171.6523]	[127.8108, 181.6523]
13	[154.5852, 164.5852]	[148.0385, 174.5852]	[131.5834, 184.5852]
14	[157.7212, 167.7212]	[151.6264, 177.7212]	[135.6171, 187.7212]
15	[161.1091, 171.1091]	[155.5028, 181.1091]	[139.9750, 191.1091]
16	[164.8219, 174.8219]	[159.7508, 184.8219]	[144.7507, 194.8219]
17	[168.9759, 178.9759]	[164.5035, 188.9759]	[150.0939, 198.9759]
18	[173.7798, 183.7798]	[167.0000, 193.7798]	[156.2731, 203.7798]
19	[179.6899, 189.6899]	[176.7620, 199.6899]	[163.8752, 209.6899]
20	[188.2740, 198.2740]	[186.5836, 208.2740]	[174.9169, 218.2740]

Table 3a. Focal elements of parameter ***w*** for Discretization A.

Focal element $A_{\omega,j}$	Basic probability assignment $M_w(A_{\omega,j})$
[2.1581, 2.4549]	0.1
[2.2739, 2.5683]	0.1
[2.3535, 2.6464]	0.1
[2.4183, 2.7100]	0.1
[2.4743, 2.7734]	0.1
[2.5256, 2.8427]	0.1
[2.5817, 2.9203]	0.1
[2.6464, 3.0101]	0.1
[2.7261, 3.1205]	0.1
[2.8419, 3.2809]	0.1

Table 3b. Focal elements of parameter ***w*** for Discretization B.

Focal element $A_{\omega j}$	Basic probability assignment $M_w(A_{\omega j})$
[2.1118, 2.4095]	0.05
[2.1936, 2.4897]	0.05
[2.2500, 2.5449]	0.05
[2.2958, 2.5898]	0.05
[2.3354, 2.6286]	0.05
[2.3708, 2.6633]	0.05
[2.4031, 2.6949]	0.05
[2.4330, 2.7254]	0.05
[2.4601, 2.7570]	0.05
[2.4873, 2.7900]	0.05
[2.5127, 2.8247]	0.05
[2.5390, 2.8612]	0.05
[2.5670, 2.9000]	0.05
[2.5969, 2.9414]	0.05
[2.6292, 2.9862]	0.05
[2.6646, 3.0352]	0.05
[2.7042, 3.0901]	0.05
[2.7500, 3.1536]	0.05
[2.8063, 3.2317]	0.05
[2.8882, 3.3451]	0.05

Table 4a. Upper CDF calculated by means of Discretization A (column 2) and Discretization B (column 3), and their relative difference.

D_s (1)	Upper CDF, Discretization A (2)	Upper CDF, Discretization B (3)	[(2)-(3)]/(2) (%)
1.4	0.1499	0.1454	2.9729
1.6	0.6395	0.6323	1.1147
1.8	0.8653	0.8619	0.3927
2	0.9473	0.9444	0.3048

Table 4b. Lower CDF calculated by means Discretization A (column 2) and Discretization B (column 3), and their relative difference.

D_s (1)	Lower CDF, Discretization A (2)	Lower CDF, Discretization B (3)	[(2)-(3)]/(2) (%)
1.6	0.1995	0.2094	-4.9370
1.8	0.4375	0.4423	-1.1022
2	0.5721	0.5782	-1.0636
3	0.9102	0.9128	-0.2808
4	0.9633	0.9632	0.0089
6	0.9907	0.9904	0.0285